

WEAK COMPACTNESS IN BANACH LATTICES

CONSTANTIN P. NICULESCU

INTRODUCTION

In his remarkable paper devoted to the study of the weakly compact operators on $C(S)$ -spaces, Grothendieck [5] has proved that a bounded subset K of $C(S)'$ is relatively weakly compact if and only if $\limsup_{n \rightarrow \infty} \sup_{\lambda \in K} |\lambda(O_n)| = 0$ for each sequence of pairwise disjoint open subsets O_n of S . This result was used later by H. P. Rosenthal [16] to formulate the notion of a relatively disjoint family of measures, which is instrumental in his criterion of weak compactness in a space $L_1(\mu)$: A bounded subset K of $L_1(\mu)$ which is not weakly compact contains a basic sequence $(x_n)_n$ which is equivalent to the natural basis of ℓ_1 and such that $\overline{\text{Span}(x_n)_n}$ is complemented in $L_1(\mu)$.

The main result of our first section extends this criterion for weakly sequentially complete Banach lattices. Alternatively, we obtain the following dichotomy theorem for operators T given on Banach spaces and taking values in weakly sequentially complete Banach lattices: T is either weakly compact or its restriction to a complemented subspace which is isomorphic to ℓ_1 is an isomorphism.

Section 2 is devoted to characterizing the behaviour of weakly compact operators defined on Banach lattices in terms of unconditional basic sequences. For example, it is proved that every operator T defined on a Banach lattice which contains no lattice isomorph of ℓ_1 is weakly compact provided that T maps weak Cauchy sequences into norm convergent sequences (the reciprocal Dunford-Pettis property). A special attention is paid to finding the conditions under which the following dichotomy result (a variation of the Dieudonné property) holds: Each operator T from E to a Banach space F is either weakly compact or the restriction of T to a subspace isomorphic to c_0 is an isomorphism.

In the present paper we shall often make use of the classical results due to Kakutani concerning the AM and AL-spaces. Particularly we shall need the following constructions of such spaces.

Let E be a Banach lattice. For each $x \in E$, $x > 0$, we can consider the order ideal generated by x :

$$E_x = \{y \in E; (\exists) \lambda > 0, |y| \leq \lambda x\}$$

normed by

$$\|y\|_x = \inf\{\lambda; |y| \leq \lambda x\}.$$

Then E_x is an AM-space with a strong order unit (which is x) and thus lattice isometric to a space $C(S_x)$. We shall denote by $i_x: E_x \rightarrow E$ the canonical inclusion.

For each $x' \in E'$, $x' > 0$, we can consider the following relation of equivalence on E

$$x \sim 0 \text{ if and only if } x'(|x|) = 0.$$

Then the completion of E/\sim with respect to the norm

$$\|x\|_{x'} = x'(|x|)$$

is an AL-space, say $L_1(x')$, and the canonical mapping $E \rightarrow L_1(x')$ will be denoted by $j_{x'}$. An useful remark is that $(j_{x'})' = i_{x'}$ for each positive $x' \in E'$.

The main results of our paper, except for Theorem 1.7, have been announced at the Symposium on Functional Analysis and its Applications, Craiova, October 28–29, 1977; see [11]. Theorem 2.1 below was also included (without proof) in [13].

The characterization given here for Banach lattices having the reciprocal Dunford-Pettis property was discovered independently by B. Kühn (Dortmund).

The author is much indebted to Ju. A. Abramovich and N. Popa for many valuable suggestions and criticism.

1. WEAK COMPACTNESS IN WEAKLY SEQUENTIALLY COMPLETE BANACH LATTICES

An important result due to H. P. Rosenthal [15] asserts that each operator T from a Banach space E to $\ell_1(I)$ is either weakly compact or the restriction of T to a complemented subspace isomorphic to ℓ_1 is an isomorphism. The aim of this section is to discuss this result for operators with values in a weakly sequentially complete Banach lattice. The essential ingredient in our approach is a weak compactness criterion in terms of sequences of pairwise disjoint elements.

A bounded subset K of a Banach space E is said to be *relatively weakly sequentially complete* if every weak Cauchy sequence of elements of K is weakly convergent (to an element of E). A subset K of a Banach lattice E is said to be *solid* if $|y| \leq |x|$, $x \in K$ implies $y \in K$.

1.1. PROPOSITION. *Let K be a bounded convex solid subset of a Banach lattice E . Then K is relatively weakly compact if and only if K verifies the following two conditions:*

- i) K is relatively weakly sequentially complete;
- ii) K contains no sequence of pairwise disjoint elements which is equivalent to the natural basis of ℓ_1 .

The proof is a consequence of the main result in [17] (which asserts that every bounded sequence of elements of a Banach space contains a subsequence which is either weak Cauchy or equivalent to the natural basis of ℓ_1) and the following

1.2. LEMMA. *Let K be a relatively weakly sequentially complete convex solid subset of a Banach lattice E . If $(x_n)_n$ is a sequence of elements of K which is equivalent to the natural ℓ_1 -basis then there exist an increasing sequence of natural numbers $k(n)$ and a sequence of pairwise disjoint elements $d_n \in K$ which is equivalent to the natural ℓ_1 -basis and such that $|d_n| \leq |x_{k(n)}|$, $n \geq 1$.*

Proof. By Lemma 1.2 in [8] we may assume that E is separable. Then E' contains strictly positive functionals φ i.e.,

$$x \geq 0, \varphi(x) = 0 \quad \text{implies} \quad x = 0.$$

In this case the canonical mapping j_φ is one-to-one, so we can identify E as a vector subspace of $L_1(\varphi)$.

The remainder of the proof will be covered in four steps.

STEP 1. If φ is strictly positive then K is an order ideal of $L_1(\varphi)$ i.e.,

$$x \in L_1(\varphi), y \in K \text{ and } |x| \leq |y| \text{ implies } x \in K.$$

Indeed, it suffices to consider the case when $x, y \geq 0$. Since E is dense in the Banach lattice $L_1(\varphi)$, there exists a sequence of elements $x_n \in E$ such that $\|x_n - x\|_\varphi \rightarrow 0$. On the other hand

$$|(x_n \vee 0) \wedge x - x| \leq |x_n \vee 0 - x| \leq |x_n - x|,$$

and thus we can assume in addition that $0 \leq x_n \leq x$ and $\|x_n - x\|_\varphi \leq 2^{-n}$ for each $n \geq 1$. Put

$$z_n = \sup_{k \geq n} x_k$$

where the supremum is taken in $L_1(\varphi)$. The sequence $(\sup_{n \leq k \leq m+n} x_k)_m$, formed by elements of K is weakly convergent, and thus $z_n \in E$. Since $0 \leq z_n \leq y$ it follows that $z_n \in K$ and in a similar way we can conclude that $z = \inf z_n \in K$. Then

$$\|x - z\|_\varphi = \lim_{n \rightarrow \infty} \|x - \sup_{k \geq n} x_k\|_\varphi \leq \lim_{n \rightarrow \infty} \|x - x_n\|_\varphi = 0$$

which implies that $x = z$.

STEP 2. If $\varphi \leq \psi$ are two strictly positive functionals then the identity of E extends to a mapping $L_1(\psi) \rightarrow L_1(\varphi)$ whose restriction to the closure \overline{K} of K in $L_1(\psi)$ is one-to-one.

In fact, we have to prove that

$$x \in \overline{K}, x > 0 \text{ implies } \varphi(x) > 0.$$

For, choose a sequence of elements $x_n \in K$ with $0 \leq x_n \leq x$ and $\|x_n - x\|_\psi \leq 2^{-n}$ for each $n \in \mathbf{N}$. As above we can conclude that

$$\|x - \inf_{k \geq n} x_k\|_\psi \rightarrow 0.$$

Because the sequence of elements $z_n = \inf_{k \geq n} x_k \in K$ is nondecreasing and $\psi(x) > 0$, it follows that $z_n > 0$ for $n \geq n_0$ and thus $\varphi(x) \geq \varphi(z_{n_0}) > 0$.

STEP 3. There exist a strictly positive functional $\varphi \in E'$ and a subsequence of $(x_n)_n$ with no weakly convergent subsequences in $L_1(\varphi)$.

In fact, if the contrary is true, then given a strictly positive functional $\varphi \in E'$, each subsequence of $(x_n)_n$ has a weakly convergent subsequence in $L_1(\varphi)$. Suppose that $x_{p(n)} \rightarrow x$ in the weak topology of $L_1(\varphi)$. For each $x' \in E'$, $x' > 0$, the functional $\tilde{\varphi} = \varphi \vee x'$ is also strictly positive, so that $(x_{p(n)})_n$ has a subsequence, say $(y_n)_n$, which is weakly convergent to y in $L_1(\tilde{\varphi})$. The result obtained at Step 2 shows that $y = x \in L_1(\varphi)$; because of the continuity of the canonical mapping $L_1(\tilde{\varphi}) \rightarrow L_1(x')$ it follows that the sequence $(x_{p(n)})_n$ must be weak Cauchy in E . Or, the natural basis of ℓ_1 has no weak Cauchy subsequence.

STEP 4. The selection of the sequences $(k(n))_n$ and $(d_n)_n$.

Let φ be a strictly positive functional as indicated at Step 3 i.e., such that the sequence $(x_n)_n$ (possibly a subsequence of it) has no weakly convergent subsequence in $L_1(\varphi)$. Since E is supposed to be separable, the Banach lattice $L_1(\varphi)$ can be viewed as an $L_1(\mu)$ -space for μ a suitable positive Radon measure on a compact Hausdorff space S . Then a well known result due to Grothendieck yields the existence of a sequence of pairwise disjoint open subsets $D_n \subset S$ and a subsequence $(x_{k(n)})_n$ of $(x_n)_n$ such that

$$\inf_{n \in \mathbf{N}} \int_{D_n} |x_{k(n)}| d\mu = \alpha > 0.$$

See [5], Theorem 2, for details.

According to the result obtained at Step 1, $d_n = x_{k(n)} \cdot \chi_{D_n} \in K$. Then $|d_m| \wedge |d_n| = 0$ ($m \neq n$) and for every finite family of real numbers λ_n we have

$$\|\sum \lambda_n d_n\| = \|\sum |\lambda_n| \cdot |d_n|\| \geq \|\varphi\|^{-1} \cdot \sum |\lambda_n| \varphi(|d_n|) \geq \alpha \|\varphi\|^{-1} \cdot \sum |\lambda_n|$$

which implies that the sequence $(d_n)_n$ is equivalent to the natural ℓ_1 -basis.

Q.E.D.

We need also a combinatorial result due to H. P. Rosenthal [16]. The argument is so elegant and short that we include it here:

1.3. LEMMA. *Let $(a_{i,j})_{i,j}$ be an infinite matrix of positive numbers such that*

$$\sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} a_{i,j} < \infty.$$

Then for each $\varepsilon > 0$ there exists an infinite subset $N_\varepsilon \subset \mathbb{N}$ such that

$$\sup_{i \in N_\varepsilon} \sum_{\substack{j \in N_\varepsilon \\ j \neq i}} a_{i,j} < \varepsilon.$$

Proof. We can choose by induction, for $i = 1, 2, \dots$, positive integers k_i and infinite subsets N_i of positive integers such that for all i :

- a) If $i > 1$, then $k_i \in N_{i-1}$, $N_{i-1} \supset N_i$ and $k_{i-1} < k_i$;
- b) $\sum_{j>l} a_{k_i,j} \leq 2^{-i}$ for all $l \in N_i$;
- c) $a_{i,k_i} \leq 2^{-i}$ for all $l \in N_i$.

Then $N_\varepsilon = \{k_p, k_{p+1}, \dots\}$ for $p = \lceil -\lg_2 \varepsilon \rceil + 2$.

Q.E.D.

We can now formulate the following criterion of weak compactness which extends previous results due to Grothendieck [5] and H. P. Rosenthal [16]:

1.4. THEOREM. *Let K be a bounded subset of a Banach lattice E , whose convex solid hull is relatively weakly sequentially complete. The following assertions are equivalent:*

- (w) *K is relatively weakly compact;*
- (R) *There is no sequence of elements $x_n \in K$ which is equivalent to the natural basis of ℓ_1 and such that $\text{Span}(x_n)_n$ is complemented in E ;*
- (G') *If $(x'_n)_n$ is a weakly summable sequence of pairwise disjoint positive elements of E' then*

$$\lim_{n \rightarrow \infty} x'_n(x) = 0$$

uniformly for $x \in K$;

- (OG') *If $(x'_n)_n$ is an order bounded sequence of pairwise disjoint positive elements of E' then*

$$\lim_{n \rightarrow \infty} x'_n(x) = 0$$

uniformly for $x \in K$.

Proof. Clearly, (w) \Rightarrow (R) and (G') \Rightarrow (OG').

(R) \Rightarrow (G'). In fact, if the contrary is true, then there are a weakly summable sequence of pairwise disjoint normalized elements $x'_n \in E'$, $x'_n > 0$, and a sequence of elements $x_n \in K$ such that $\inf |x'_n(x_n)| = \delta > 0$. Put $\lambda = \sup\{\|x\|; x \in K\}$, $\mu = \sup\{\sum |x''(x'_n)|; x'' \in E'', \|x''\| \leq 1\}$ and $\varepsilon = (1/2)\inf\{\delta, \delta^2 \lambda^{-1} \mu^{-1}\}$. By Lemma 1.3 we can suppose in addition that

$$\sum_{i \neq n} |x'_i(x_n)| < \varepsilon, \quad n \geq 1.$$

Then for every finite family of scalars λ_n and for $\varepsilon_n = \text{sign} \lambda_n x'_n(x_n)$ we obtain that

$$\begin{aligned} \mu \lambda \sum |\lambda_n| &\geq \mu \|\sum \lambda_n x_n\| \geq |\langle \sum \lambda_n x_n, \sum \varepsilon_i x'_i \rangle| \geq \\ &\geq \delta \sum |\lambda_n| - \sum_n \sum_{i \neq n} |\lambda_n| \cdot |x'_i(x_n)| \geq \frac{\delta}{2} \sum |\lambda_n|, \end{aligned}$$

which implies that the sequence $(x_n)_n$ is equivalent to the natural basis of ℓ_1 . Let P_n be the band projection generated by x'_n in E' . The functionals $d_n(\cdot) = P_n(\cdot)|_{x_n} \in E''$ are pairwise disjoint, $0 < d_n \leq |x_n|$ and in addition

$$d_n(x'_n) \geq \delta \text{ and } d_n(x'_m) = 0 \text{ for } m \neq n,$$

which implies that the sequence $(d_n)_n$ is equivalent to the natural basis of ℓ_1 . Moreover, there is a positive projection Q of E'' onto $\overline{\text{Span}(d_n)}$ given by

$$Q(x'') = \sum_{n=1}^{\infty} \frac{x''(x'_n)}{d_n(x'_n)} d_n.$$

The isomorphism $U: \overline{\text{Span}(x_n)}_n \rightarrow \overline{\text{Span}(d_n)}_n$ given by $U(x_n) = \frac{x'_n(x_n)}{d_n(x'_n)} d_n, n \geq 1$, verifies $\|Ux - Qx\| \leq \frac{1}{2} \|Ux\|$ for all $x \in \overline{\text{Span}(x_n)}_n$. Consequently the operator $P = (Q|_{\overline{\text{Span}(x_n)}_n})^{-1} \circ Q$ provides a projection of E'' onto $\overline{\text{Span}(x_n)}_n$.

(OG') \Rightarrow (w). Without loss of generality we can suppose that K is also convex and solid. If K is not relatively weakly compact, then by Proposition 1.1 above there exists a sequence of pairwise disjoint elements d_n of K which is equivalent to the natural basis of ℓ_1 . For the sake of convenience we shall identify $\overline{\text{Span}(d_n)}_n$ as ℓ_1 . Let $u' \in E'$ a positive extension of $\mathbf{1} = (1, 1, \dots) \in \ell_\infty$ to E . Put $d'_n = u' \circ [d_n]|_E, n \geq 1$, where $[d_n]$ denotes the band projection generated by d_n in E'' . For each $x \in E'', x > 0$, we have

$$\sum d'_n(x) \leq u'(x)$$

so that the sequence $(d'_n)_n$ is weakly summable and formed by pairwise disjoint functionals. But $d'_n(d_n) = 1$ for each $n \in \mathbb{N}$, in contradiction with (OG').

Q.E.D.

1.5. COROLLARY. *Let K be a bounded subset of a weakly sequentially complete Banach lattice E . Then:*

- i) *K is relatively weakly compact if and only if K satisfies the following condition*
 \ll *For each order bounded sequence $(x'_n)_n$ of pairwise disjoint positive elements of E' we have*

$$\lim_{n \rightarrow \infty} x'_n(x) = 0 \text{ uniformly for } x \in K \gg;$$

ii) If K is relatively weakly compact so is its convex solid hull.

A result due to Lozanovskii asserts that a Banach lattice E is weakly sequentially complete if and only if no closed sublattice of E is lattice isomorphic to e_0 . See also [7] and [8]. Consequently the assertion i) in Corollary 1.5 above is equivalent to the fact that E is weakly sequentially complete.

A first dichotomy result for operators having values in a weakly sequentially Banach lattice is indicated by the following :

1.6. PROPOSITION. *Let T be a bounded linear operator from a Banach space E to a weakly sequentially complete Banach lattice F . Then T is either weakly compact or the restriction of T to a complemented subspace X , which is isomorphic to ℓ_1 , is an isomorphism.*

This result is due to H. P. Rosenthal [15] for $F = \ell_1(\Gamma)$. The proof for F an arbitrary weakly sequentially complete Banach lattice follows the same line and depends on Theorem 1.4 (G') above.

The result of Proposition 1.6 is also valid for F a Banach space whose dual is a C^* -algebra. In this case the proof makes use of Theorem II.2 in [2].

If E is a Banach lattice and $T \geq 0$ then X can be chosen to be a sublattice.

We need the fact (due independently to Abramovich and Lotz) that ℓ_1 is complemented in any Banach lattice E which contains it as a sublattice. A short argument is as follows: At first notice that for each closed sublattice $\overline{\text{Span}(d_n)} \subset E$ which is lattice isomorphic to ℓ_1 there exists a positive functional $x' \in E'$ with $\inf x'(d_n) > 0$. Then $\overline{\text{Span}(j_{x'}(d_n))}$ is a closed sublattice of $L_1(x')$ and thus complemented in $L_1(x')$. If Q denotes a positive projection of $L_1(x')$ onto $\overline{\text{Span}(j_{x'}(d_n))}$ then $P = (j_{x'})^{-1} \circ Q \circ j_{x'}$ provides a positive projection of E onto $\overline{\text{Span}(d_n)}$.

1.7. THEOREM. *Let T be a positive operator from a Banach lattice E to a weakly sequentially complete Banach lattice F . Then T is either weakly compact or the restriction of T to a complemented sublattice of E which is lattice isomorphic to ℓ_1 is an isomorphism.*

Proof. Suppose that T is not weakly compact. Then by Corollary 1.5 above there exist an $\alpha > 0$, a positive $\varphi' \in F'$ and a sequence $(\varphi'_n)_n$ of pairwise disjoint elements of F' such that

$$0 < \varphi'_n \leq \varphi'$$

and

$$\|T'\varphi'_n\| > \alpha \quad \text{for all } n \in \mathbf{N}.$$

Since the sequence $(\varphi'_n)_n$ is equivalent to the natural basis of e_0 and $\inf \|T'\varphi'_n\| > 0$, we may assume (by passing to a subsequence if necessary) that $T' \overline{\text{Span}(\varphi'_n)}$ is an isomorphism. (See Remark 1 to Theorem 3.4 in [16].) Put $d'_n = T'\varphi'_n$, $n \in \mathbf{N}$.

Then $\|d'_n\| > \alpha$, $0 < d'_n \leq T'\varphi'$ and

$$\|d'_1 - d'_2 - \dots - d'_n\| \leq C$$

for all $n \in \mathbb{N}$.

According to Proposition 1 in [9] there exist a sequence $(\tilde{d}'_n)_n$ of pairwise disjoint positive elements of E' and a sequence $(k(n))_n \subset \mathbb{N}$ such that

$$\|\tilde{d}'_n\| > \alpha \text{ and } 0 < \tilde{d}'_n \leq d'_{k(n)},$$

for all $n \in \mathbb{N}$. Again, by passing to a subsequence, we may assume that $k(n) = n$, $n \in \mathbb{N}$.

Let us choose for each $n \in \mathbb{N}$ a positive norm 1 element $x_n \in E$ such that $\tilde{d}'_n(x_n) \geq \alpha$ and put $x = \sum 2^{-n} x_n$. Then E_x is lattice isometric to a space $C(S_x)$ and the positive Radon measures $\mu_n = x_n \cdot \tilde{d}'_n \in C(S_x)'$ constitute a basic sequence which is equivalent to the natural basis of ℓ_1 . In fact, $\mu_m \wedge \mu_n = 0$ for $m \neq n$ and $\|\mu_n\| = \mu_n(1) = \tilde{d}'_n(x_n) \geq \alpha$. By Theorem 2 in [5], there exists a sequence of pairwise disjoint elements $y_n \in C(S_x)$, $0 < y_n \leq x$, such that (by passing to a subsequence if necessary)

$$\inf_{n \in \mathbb{N}} \int_{S_x} y_n d\mu_n > 0.$$

The elements $d_n = x_n \cdot y_n \in C(S_x) \subset E$ are pairwise disjoint, $\|d_n\| \leq 1$ for all $n \in \mathbb{N}$, and $\inf d'_n(d_n) = \gamma > 0$.

By Lemma 1.3 above we may assume in addition that

$$\sup_{n \in \mathbb{N}} \sum_{k \neq n} d'_k(d_n) < \frac{\gamma}{2}.$$

Put

$$\delta = \sup\{\sum |x''(d'_n)|; x'' \in E'', \|x''\| \leq 1\}.$$

Then for each finite family of real numbers λ_n we have

$$\begin{aligned} & (\delta \|T\|) \cdot \sum_n |\lambda_n| \geq \delta \|T\| \cdot \|\sum_n \lambda_n d_n\| \geq \delta \|\sum_n \lambda_n T d_n\| \geq \\ & \geq |\langle \sum_n \lambda_n T d_n, \sum_k (\text{sign } \lambda_k) \cdot \varphi'_k \rangle| \geq \sum_n |\lambda_n| \cdot (T'\varphi'_n) d_n - \sum_n \sum_{k \neq n} |\lambda_n| \cdot (T'\varphi'_k) d_n \geq \frac{\gamma}{2} \sum |\lambda_n|, \end{aligned}$$

which implies that $X = \overline{\text{Span}(d_n)}$ is lattice isomorphic to ℓ_1 and $T|X$ is an isomorphism.

Q.E.D.

The following corollary precises a well known result due to Bessaga and Pelczyński [3]:

1.8. COROLLARY. *Let E be a Banach lattice. Then E' contains a lattice isomorph of ϵ_0 if and only if E contains a lattice isomorph of ℓ_1 .*

Proof. At first suppose there exists a sequence $(e'_n)_n$ of pairwise disjoint positive elements of E' which is equivalent to the natural basis of ϵ_0 . Then the mapping $T: E \rightarrow \ell_1$ given by $Tx = (e'_n(x))_n$ for all $x \in E$, satisfies the hypotheses of Theorem 1.7 above and thus E contains a lattice isomorph of ℓ_1 .

The other implication is due to H. P. Lotz. (See [7], Proposition 2.6, for details.)

Q.E.D.

The role of ϵ_0 and ℓ_1 in Corollary 1.8 above cannot be inverted. In fact, the Banach lattice $E = \left(\sum_{n=1}^{\infty} \oplus \ell_{\infty}(n) \right)_{\ell_1}$ is weakly sequentially complete though E' contains ℓ_1 as a sublattice.

1.9. COROLLARY. *The following assertions are equivalent for E a Banach lattice:*

- i) *E contains no complemented isomorph of ℓ_1 ;*
- ii) *E contains no lattice isomorph of ℓ_1 ;*
- iii) *E' is weakly sequentially complete.*

The proof of Theorem 1.7 above yields also the following

1.10. COROLLARY. *Let E a Banach lattice and let $(d'_n)_n$ a bounded sequence of pairwise disjoint elements of E' which is equivalent to the natural basis of ℓ_1 . Then there exists a sequence $(d_n)_n$ of pairwise disjoint positive elements of E such that*

$$\sup \|d_n\| \leq 1 \text{ and } \inf |d'_n(d_n)| > 0.$$

The sequence $(d_n)_n$ need not necessarily be equivalent to the natural basis of ϵ_0 . Here is an example: Let us denote by $(u_n)_n$ and $(v_n)_n$ the natural bases of ϵ_0 and respectively ℓ_2 and let $(u'_n)_n$ and $(v'_n)_n$ be their biorthogonal sequences. Consider the Banach lattice $E = \epsilon_0 \oplus \ell_2$ and put $w_n = \frac{1}{2}u_n \oplus v_n \in E$ and $w'_n = \frac{1}{2}u'_n \oplus v'_n \in E'$ for each $n \in \mathbb{N}$. Then $(w_n)_n$ and $(w'_n)_n$ are sequences of pairwise disjoint positive elements, $w'_n(w_n) = 1$ for each $n \in \mathbb{N}$, $(w_n)_n$ is equivalent to the natural basis of ℓ_2 and $(w'_n)_n$ is equivalent to the natural basis of ℓ_1 .

1.11. PROBLEM. *Let E be a Banach lattice which contains no lattice isomorph of ℓ_1 . Can the sequence $(d_n)_n$ in Corollary 1.10 above be chosen as being equivalent to the natural basis of ϵ_0 ?*

Two special cases of this problem will be treated in Theorem 2.4 below.

We end this section with another problem concerning the position of ℓ_1 among the subspaces of a Banach lattice:

1.12. PROBLEM. *Let E be a weakly sequentially complete Banach lattice and let $(x_n)_n$ be a sequence of elements of E which is equivalent to the natural basis of ℓ_1 . Does there exist a projection of E onto $\overline{\text{Span}(x_n)}$?*

The answer is unknown even for E an $L_1(\mu)$ -space.

Our Theorem 1.4 (R) above shows that each subsequence of $(x_n)_n$ has a subsequence whose closed linear span is complemented in E .

2. THE RECIPROCAL DUNFORD-PETTIS PROPERTY AND OTHER DISTINGUISHED PROPERTIES

According to [5] a Banach space E is said to have the *reciprocal Dunford-Pettis property* if every operator T from E into an arbitrary Banach space F is weakly compact provided that T maps weakly convergent sequences into norm convergent sequences. The classical examples of such spaces are $C(S)$ spaces and reflexive Banach spaces. The reciprocal Dunford-Pettis property is stable under finite products and passing to quotients.

2.1. THEOREM. *A Banach lattice E has the reciprocal Dunford-Pettis property if and only if E contains no complemented isomorph of ℓ_1 , if and only if E contains no lattice isomorph of ℓ_1 .*

Particularly we obtain some few more Banach spaces with the reciprocal Dunford-Pettis property such as

$$\left(\sum_{n=1}^{\infty} \oplus \ell_p(n) \right)_{\epsilon_0}, \quad 1 \leq p < \infty.$$

The proof of Theorem 2.1 is an immediate consequence of Corollary 1.9 and the following

2.2. LEMMA. *Let E be a Banach lattice which contains no lattice isomorph of ℓ_1 and let K be a bounded subset of E' such that*

$$(rDP) \quad \lim_{n \rightarrow \infty} \sup_{x' \in K} |x'(x_n)| = 0$$

for each weakly convergent sequence of pairwise disjoint elements $x_n \in E$.

Then K is relatively weakly compact.

Proof. At first notice that (rDP) holds also for the convex solid hull \tilde{K} of K . We shall show that \tilde{K} is relatively weakly compact. In fact, if the contrary is true, then by Corollary 1.9 and Proposition 1.1 there exists a sequence $(d'_n)_n$ of pairwise disjoint positive elements of \tilde{K} which is equivalent to the natural basis of ℓ_1 . An appeal to Corollary 1.10 yields also a sequence $(d_n)_n$ of pairwise disjoint positive elements of E such that

$$\|d_n\| \leq 1 \quad \text{and} \quad \inf d'_n(d_n) > 0.$$

To conclude it suffices to show that $d_n \rightarrow 0$ in the weak topology of E . The argument given here is due to Abramovich. (See [1], Lemma 3.1.)

If $(d_n)_n$ is not weakly convergent to 0, then there exists an $x' \in E', x' > 0$, such that $\inf x'(d_n) = \alpha > 0$. Then for every family of real numbers λ_n we have

$$\sum |\lambda_n| \geq \| \sum \lambda_n d_n \| = \| \sum |\lambda_n| d_n \| \geq \frac{1}{\|x'\|} \sum |\lambda_n| x'(d_n) \geq \frac{\alpha}{\|x'\|} \sum |\lambda_n|$$

in contradiction with the fact that E contains no lattice isomorph of ℓ_1 .

Q.E.D.

Another distinguished property introduced in [5] is the so called *Dieudonné property*: a Banach space E is said to have the Dieudonné property if every operator T from E into an arbitrary Banach space F is weakly compact provided that T maps weak Cauchy sequences into weakly convergent sequences. In the sequel we shall be concerned with a slightly stronger condition of weak compactness related to earlier work of Grothendieck [5] and Pełczyński [14] on $C(S)$ spaces:

2.3. DEFINITION. A Banach space E has the *strict Dieudonné property* if E verifies the following two equivalent conditions:

(P) Every operator T from E into a Banach space F is either weakly compact or the restriction of T to a subspace isomorphic to e_0 is an isomorphism;

(G) A bounded subset $K \subset E'$ is relatively weakly compact if (and only if) for every weakly summable sequence of elements $x_n \in E$ we have

$$\lim_{n \rightarrow \infty} x'(x_n) = 0 \quad \text{uniformly for } x' \in K.$$

It is clear that a Banach space having the strict Dieudonné property contains no complemented isomorph of ℓ_1 and it is open if this condition is also sufficient. In the context of Banach lattices this problem reduces to Problem 1.11 above.

The strict Dieudonné property is stable under finite products and passing to quotients.

2.4. THEOREM. *Let E be a Banach lattice which contains no lattice isomorph of ℓ_1 . Then each of the following conditions implies that E has the strict Dieudonné property:*

a) E is σ -complete and E has σ -continuous norm (e.g., E has an unconditional basis);

b) For each $x'' \in E''$ there exists an $x \in E$ such that $|x''| \leq x$.

Proof. It suffices to show that each bounded convex solid subset $K \subset E'$ which verifies the condition (G) in Definition 2.3 above is relatively weakly compact. In fact, if the contrary is true, then by Corollary 1.9 and Proposition 1.1 there exists a sequence of pairwise disjoint positive elements $d_n'' \in K$ which is equivalent to the

natural basis of ℓ_1 . For the sake of convenience we shall identify $\overline{\text{Span}(d'_n)_n}$ as ℓ_1 . Let $u'' \in E''$ a positive extension of $\mathbf{1} = (1, 1, \dots) \in \ell_\infty$ to E and put $z_n = u'' \circ [d'_n] \in E'' (n \in \mathbf{N})$ where $[d'_n]$ denotes the band projection generated by d'_n in E' . The sequence $(z_n)_n$ is weakly summable and formed by pairwise disjoint elements. Then the proof ends in the case a) by observing that

$$x''(x') = \sup_{\substack{0 \leq x \leq x'' \\ x \in E}} x'(x)$$

for every $x' \in E'_+$ and $x'' \in E''_+$. An alternative proof is obtained by combining our Lemma 2.2 with Lemma 2 in [18], which asserts that for each weak Cauchy sequence $(x_n)_n$ in E there exists an weakly summable sequence $(y_n)_n$ in E such that $\left(x - \sum_{k=1}^n y_k\right)_n$ is weakly convergent to 0.

In the case b) let us denote by u a positive element of E such that $u \geq u''$. Then we can proceed as in the proof of Theorem 1.7 above by choosing $x_n = u/\|u\|$ for each $n \geq 1$.

Q.E.D.

2.5. COROLLARY. *Let E be a σ -complete Banach lattice such that E' has the Schur property (i.e., every weakly convergent sequence of elements of E' is norm convergent) Then E has the strict Dieudonné property.*

The same is true for E a σ -complete Banach lattice whose dual has the Radon-Nikodym property (i.e., every absolutely summing operator from a $C(S)$ -space into E is nuclear).

Proof. Since E is σ -complete and contains no isomorph of ℓ_∞ it follows from Corollary 10 in [9] that E has order continuous norm and thus applies Theorem 2.4 a) above.

Q.E.D.

Particularly, from the result above it follows that the Banach space

$$\mathcal{J} = \left(\sum_{n=1}^{\infty} \oplus \ell_1(n) \right)_{c_0}$$

has the strict Dieudonné property. The space \mathcal{J} can be isometrically embedded into c_0 e.g., by using the mapping

$$(\alpha_n) \rightarrow (\alpha_1, \alpha_2 + \alpha_3, \alpha_2 - \alpha_3, \alpha_4 + \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 - \alpha_6, \alpha_4 - \alpha_5 + \alpha_6, -\alpha_4 + \alpha_5 + \alpha_6, \dots).$$

2.6. THEOREM. *Let E be a Banach space isomorphic to a subspace or to a quotient of c_0 . Then E has the strict Dieudonné property.*

Proof. The nontrivial case is where E is a subspace of c_0 . By Theorem 10 in [5], if $T \in L(E, F)$ is not weakly compact then there exists a weakly convergent sequence $(x_n)_n$ in E such that $\inf \|Tx_{n+1} - Tx_n\| > 0$. According to Lemma 9 in [5] the restriction of T to a subspace isomorphic to c_0 is an isomorphism.

Q.E.D.

A Banach space E is said to have the *Grothendieck property* if every weakly*-convergent sequence in E' is weakly convergent (equivalently, every operator from E into c_0 is weakly compact). Grothendieck has noted in [5] that this is the case if E is a space $C(S)$ with S a Stonean compact space.

A complete characterization in the context of Banach lattices is now given:

2.7. THEOREM. *The following assertions are equivalent for E a Banach lattice:*

- a) E has the Grothendieck property;
- b) E has the strict Dieudonné property and E contains no complemented isomorph of c_0 .

Proof. a) \Rightarrow b). We have only to verify the weak compactness criterion (G) in Definition 2.3 above. For, notice first that because E contains no complemented isomorph of ℓ_1 it follows that E' is weakly sequentially complete. Let K be a bounded subset of E' such that for each weakly summable sequence $(x_n)_n$ of elements of E we have

$$(*) \quad \lim_{n \rightarrow \infty} x'(x_n) = 0$$

uniformly for $x' \in K$. It is clear that K can be replaced by its convex solid hull, so by Proposition 1.1 it remains to prove that K contains no sequence of pairwise disjoint positive elements which is equivalent to the natural basis of ℓ_1 . In fact, if $(d'_n)_n$ is such a sequence, then

$$\overline{\lim}_{n \rightarrow \infty} d'_n(d) > 0$$

for some $d \in E, d > 0$ (E has the Grothendieck property and $(d'_n)_n$ is not weak Cauchy) and thus $(d'_n|E_d)_n$ is also equivalent to the natural basis of ℓ_1 . Since E_d is isometric to a space $C(S)$ and each $C(S)$ space has the strict Dieudonné property, the later assertion contradicts (*).

b) \Rightarrow a). Let $T \in L(E, c_0)$. Because E has the strict Dieudonné property it follows that T is either weakly compact or the restriction of T to a subspace F which is isomorphic to c_0 is an isomorphism. Because c_0 is separable injective, there exists a bounded projection P of c_0 onto $T(F)$. Then $Q = (T|F)^{-1} \circ P \circ T$ provides a bounded projection of E onto F , in contradiction with the fact that E contains no complemented isomorph of c_0 .

Q.E.D.

2.8. COROLLARY. *Let E be a Banach space such that E' has the strict Dieudonné property. Then E' has also the Grothendieck property.*

The proof is an immediate consequence of Theorem 2.7 above (actually we need only the implication $b) \Rightarrow a)$ which is true for all Banach spaces) and the fact that a dual Banach space contains no complemented isomorph of c_0 . The later can be motivated as follows: Let $(e_n)_n$ be a basic sequence in a dual Banach space F' which is equivalent to the natural basis of c_0 , let $i: \overline{\text{Span}(e_n)_n} \rightarrow F'$ the canonical inclusion and let $P: F''' \rightarrow F'$ the canonical projection given by $Pf'''(f) = f'''(f)$ for all $f \in F$ and $f''' \in F'''$. Since the restriction of $P \circ i''$ to $\overline{\text{Span}(e_n)_n}$ is an isomorphism, an appeal to Proposition 1.2 in [16] yields an infinite subset $\tilde{N} \subset \mathbb{N}$ such that $P \circ i''|[\overline{\text{Span}(e_n)_{n \in \tilde{N}}}]''$ is an isomorphism. Since c_0 is not complemented in ℓ_∞ it follows that $\overline{\text{Span}(e_n)_n}$ is not complemented in F' .

Q.E.D.

By Theorem 2.7 above it follows also that a space $C(S)$ has the Grothendieck property if and only if $C(S)$ contains no complemented isomorph of c_0 .

We end this paper with an open problem concerning the position of c_0 as a subspace of a Banach lattice:

2.9. PROBLEM. *Let E be a Banach lattice which contains no isomorphic copy of ℓ_∞ and let F be a subspace of E isomorphic to c_0 . Is F complemented in E ?*

The answer is known to be affirmative for $E = C[0, 1]$ and thus for all separable Banach spaces (Sobczyk's result on separable injectivity of c_0).

REFERENCES

1. ABRAMOVICH, JU. A., New characterizations of AM-spaces (Russian), *Analele Universității Craiova*, **6**(1978), 15–26.
2. AKEMANN, C. A., The dual space of an operator algebra, *Trans. Amer. Math. Soc.*, **126**(1967), 286–302.
3. BESSAGA, C.; PEŁCZYŃSKI, A., On bases and unconditional convergence of series in Banach spaces, *Studia Math.*, **17**(1958), 151–164.
4. DAY, M. M., *Normed linear spaces*, Springer Verlag, 1973.
5. GROTHENDIECK, A., Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, *Canad. J. Math.*, **5**(1953), 129–173.
6. LOZANOVSKII, G. IA., On ordered Banach spaces and bases (Russian), *Funktsional. Analiz i Priložen.*, **3**(1967), 92.
7. LOTZ, H. P., Minimal and reflexive Banach lattices, *Math. Ann.*, **209**(1974), 117–126.
8. MEYER-NIEBERG, P., Zur schwachen Kompaktheit in Banachverbänden, *Math. Z.*, **134**(1973), 303–315.
9. MEYER-NIEBERG, P., Charakterisierung einiger topologischer und ordnungstheoretischer Eigenschaften von Banachverbänden mit Hilfe disjunkter Folgen, *Arch. Math.*, **24**(1973), 640–647.

10. NICULESCU, C., Banach lattices with a weak order unit, *Bull. Amer. Math. Soc.*, **82**(1976), 748–750.
11. NICULESCU, C., Weak compactness in ordered Banach spaces, *INCREST Preprint Series in Math.*, n° 25/1979.
12. NICULESCU, C., A note on weakly compact operators defined on C^* -algebras, *Rev. Roumaine Math. Pures Appl.*, **25**(1980), 631–634.
13. NICULESCU, C.; VRACIU, G., On the integral representation of linear operators on $C(S)$. The significance of the Baire space, *Rev. Roumaine Math. Pures Appl.*, **25**(1980), 1407 – 1410.
14. PEŁCZYŃSKI, A., Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci.*, **12**(1962), 641–648.
15. ROSENTHAL, H. P., On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures, *Acta Math.*, **124**(1970), 205–248.
16. ROSENTHAL, H. P., On relatively disjoint families of measures with some applications to Banach space theory, *Studia Math.*, **37**(1970), 13–36.
17. ROSENTHAL, H. P., A characterization of Banach spaces containing ℓ_1 , *Proc. Nat. Acad. Sci. USA.*, **71**(1974), 2411–2413.
18. TZAFRIRI, L., Reflexivity in Banach lattices and their subspaces, *J. Functional Analysis*, **10**(1972), 1–18.

CONSTANTIN P. NICULESCU

Department of Mathematics,

University of Craiova,

Craiova 1100,

Romania.

Received April 23, 1980; revised February 5, 1981.